THE MEANING OF DIMENSIONS

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Abstract

We review the current status of dimensions, as the result of a long and controversial history that includes input from philosophy and physics. Our conclusion is that they are subjective but essential concepts which provide a kind of book-keeping device, their number increasing as required by advances in physics. The world almost certainly has more than the four dimensions of space and time, but the introduction of the fifth and higher dimensions requires a careful approach wherein known results are embedded and new ones are couched in the most productive manner.

1   <u>Introduction</u>

Dimensions are both primitive concepts that provide a framework for mechanics and sophisticated devices that can be used to construct unified field theories. Thus the ordinary space of our perceptions (xyz) and the subjective notion of time (t) provide the labels with which to describe Newtonian mechanics, or with the introduction of the speed of light to form an extra coordinate (ct) the mechanics of 4D Einstein relativity. But used in the ab-

stract, they also provide a means of extending general relativity in accordance with certain physical principles, as in 10D supersymmetry. As part of the endeavour to unify gravity with the interactions of particle physics, there has recently been an explosion of interest in manifolds with higher dimensions. Much of this work is algebraic in nature, and has been reviewed elsewhere (see below). Therefore, to provide some balance and direction, we will concentrate here on fundamentals and attempt to come to an understanding of the meaning of dimensions.

Our main conclusion, based on 35 years of consideration, will be that dimensions are basically inventions, which have to be chosen with skill if they are to be profitable in application to physics

This view may seem strange to some workers, but is not new. It is implicit in the extensive writings on philosophy and physics by the great astronomer Eddington, and has been made explicit by his followers, who include the writer. This view is conformable, it should be noted, with algebraic proofs and other mathematical results on many-dimensional manifolds, such as those of the classical geometer Campbell, whose embedding theorem has been recently rediscovered and applied by several workers to modern unified-field theory. Indeed, a proper understanding of the meaning of dimensions

involves both history and modern physics.

There is a large literature on dimensions; but it would be inappropriate to go into details here, and we instead list some key works. The main philosophical/physical ones are those by Barrow (1981), Barrow and Tipler (1986), Eddington (1935, 1939), Halpern (2004), Kilmister (1994), McCrea and Rees (1983), Petley (1985), Price and French (2004) and Wesson (1978, 1992). The main algebraic/mathematical works are those by Campbell (1926), Green et al. (1987), Gubser and Lykken (2004), Seahra and Wesson (2003), Szabo (2004), Wesson (1999, 2006) and West (1986). These contain extensive bibliographies, and we will quote freely from them in what follows.

The plan of this paper is as follows. Section 2 outlines the view of our group, that dimensions are inventions whose geometrical usefulness for physics involves a well-judged use of the fundamental constants. This rests on work by Eddington, Campbell and others; so in Sections 3 and 4 we give accounts of the main philosophical and algebraic results (respectively) due to these men, in a modern context. Section 5 is a summary, where we restate our view that the utility of dimensions in physics owes at least as much to skill as to symbolism. We aim to be pedagogical rather than pedantic, and hope that the reader will take our comments in the spirit of learning rather

than lecture.

2 <u>Dimensions and Fundamental Constants</u>

Minkowski made a penetrating contribution to special relativity and our view of mechanics when by the simple identification of $x^4 \equiv ct$ he put time on the same footing as the coordinates $(x^{123} = xyz)$ of the ordinary space of our perceptions. Einstein took an even more important step when he made the Principle of Covariance one of the pillars of general relativity, showing that the 4 coordinates traditionally used in mechanics can be altered and even mixed, producing an account of physical phenomena which is independent of the labels by which we choose to describe them. These issues are nowadays taken for granted; but a little reflection shows that insofar as the coordinates are the labels of the dimensions, the latter are themselves flexible in nature.

Einstein was in his later years also preoccupied with the manner in which we describe matter. His original formulation of general relativity involved a match between a purely geometrical object we now call the Einstein tensor $(G_{\alpha\beta}, \alpha$ and $\beta = 0, 123$ for $t, xyz)$, and an object which depends on the properties of matter which is known as the energy-momentum (or stress-energy)

tensor ($T_{\alpha\beta}$, which contains quantities like the ordinary density ρ and pressure p of matter). The coefficient necessary to turn this correspondence into an equation is (in suitable units) $8\pi G\!\left/c^4\right.$, where G is the gravitational constant. Hence, Einstein's field equations, $G_{\alpha\beta} = \left(8G\!\left/c^4\right.\right)T_{\alpha\beta}$, which are an excellent description of gravitating matter. In writing these equations, it is common to read them from left to right, so that the geometry of 4D spacetime is governed by the matter it contains. However, this split is artificial. Einstein himself realized this, and sought (unsuccessfully) for some way to turn the "base wood" of $T_{\alpha\beta}$ into the "marble" of $G_{\alpha\beta}$. His aim, simply put, was to geometrize all of mechanical physics - the matter as well as the fields.

A potential way to geometrize the physics of gravity and electromagnetism was suggested in 1920 by Kaluza, who added a fifth dimension to Einstein's general relativity. Kaluza showed in essence that the apparently empty 5D field equations $R_{AB} = 0\,(A, B = 0, 123, 4)$ in terms of the Ricci tensor, contain Einstein's equations for gravity and Maxwell's equations for electromagnetism. Einstein, after some thought, endorsed this step. However, in the 1920s quantum mechanics was gaining a foothold in theoretical physics, and in the 1930s there was a vast expansion of interest in this area, at the expense of general relativity. This explains why there was such a

high degree of attention to the proposal of Klein, who in 1926 suggested that the fifth dimension of Kaluza ought to have a closed topology (i.e., a circle), in order to explain the fundamental quantum of electric charge (l). Klein's argument actually related this gravity to the momentum in the extra dimension, but in so doing introduced the fundamental unit of action (h)which is now known as Planck's constant. However, despite the appeal of Klein's idea, it was destined for failure. There are several technical reasons for this, but it is sufficient to note here that the crude 5D gravity/quantum theory of Kaluza/Klein implied a basic role for the mass quantum $(h\,c\diagup G)^{1/2}$. This is of order 10^{-5} g, and does <u>not</u> play a dominant role in the spectrum of masses observed in the real universe. (In more modern terms, the so-called hierarchy problem is centred on the fact that observed particle masses are far less than the Planck mass, or any other mass derivable from a tower of states where this is a basic unit.) Thus, we see in retrospect that the Klein modification of the Kaluza scheme was a dead-end. This does not, though, imply that there is anything wrong with the basic proposition, which follows from the work of Einstein and Kaluza, that matter can be geometrized with the aid of the fundamental constants. As a simple example, an astrophysicist presented with a problem involving a gravitationally-dominated cloud of

density ρ will automatically note that the free-fall or dynamical timescale is the inverse square root of $G\rho$. This tells him immediately about the expected evolution of the cloud. Alternatively, instead of taking the density as the relevant physical quantity, we can form the length $(c^2/G\rho)^{1/2}$ and obtain an equivalent description of the physics in terms of a geometrical quantity.

The above simple outline, of how physical quantities can be combined with the fundamental constants to form geometrical quantities such as lengths, can be much developed and put on a systematic basis (Wesson 1999). The result is induced-matter theory, or as some workers prefer to call it, space-time-matter theory. The philosophical basis of the theory is to realize Einstein's dream of unifying geometry and matter (see above). The mathematical basis of it is Campbell's theorem, which ensures an embedding of 4D general relativity with sources in a 5D theory whose field equations are apparently empty (see below). That is, the Einstein equations $G_{\alpha\beta} = (8\pi G/c^4) T_{\alpha\beta} (\alpha, \beta = 0, 123)$ are embedded perfectly in the Ricci-flat equations $R_{AB} = 0 \, (A, B = 0, 123, 4)$. The point is, in simple terms, that we use the fifth dimension to model matter.

An alternative version of 5D gravity, which is mathematically similar, is membrane theory. In this, gravity propagates freely in 5D, into the "bulk";

but the interactions of particles are confined to a hypersurface or the "brane". It has been shown by Ponce de Leon and others that both the field equations and the dynamical equations are effectively the same in both theories. The only difference is that whereas induced-matter theory treats all 5 dimensions as equivalent, membrane theory makes spacetime a special (singular) hypersurface. For induced-matter theory, particles can wander away from the hypersurface at a slow rate governed by the cosmological constant; whereas for membrane theory, particles are confined to the hypersurface by an exponential force governed by the cosmological constant. Both versions of 5D general relativity are in agreement with observations. The choice between them is largely philosophical: Are we living in a universe where the fifth dimension is "open", or are we living an existence where we are "stuck" to a particular slice of 5D manifold?

Certainly, the fundamental constants available to us at the present stage in the development of physics allow us to geometrize matter in terms of <u>one</u> extra dimension. Insofar as mechanics involves the basic physical quantities of mass, length and time, it is apparent that any code for the geometrization of mass will serve the purpose of extending 4D spacetime to a 5D space-time-mass manifold (the theory is covariant). However, not all parametizations

are equally convenient, in regard to returning known 4D physics from a 5D definition of "distances" or metric. Thus, the "canonical" metric has attracted much attention. In it, the line element is augmented by a flat extra dimension, while its 4D part is multiplied by a quadratic factor (the corresponding metric is membrane theory involves an exponential factor, as noted above). The physics flows from this factor, which is $(l/L)^2$ where $x^4 = l$ and L is a constant which by comparison with the 4D Einstein metric means $L = (3/\Lambda)^{1/2}$ where Λ is the cosmological constant. In this way, we weld ordinary mechanics to cosmology, with the identification $x^4 = l = Gm/c^2$ where m is the rest mass of a macroscopic object. If, on the other hand, we wish to study microscopic phenomena, the simple coordinate transformation $l \rightarrow L^2/l$ gives us a quantum (as opposed to classical) description of rest mass via $x^4 = h/mc$. In other words, the large and small scales are accommodated by choices of coordinates which utilize the available fundamental constants, labelling the mass either by Schwarzchild radius or by the Compton wavelength.

It is not difficult to see how to extend the above approach to higher dimensions. However, skill is needed here. For example, electric charge can either be incorporated into 5D (along the lines originally proposed by Kaluza and

Klein), or treated as a sixth dimension (with coordinate $x_q \equiv (G/c^4)^{1/2} q$ where q is the charge, as studied by Fukui and others). A possible resolution of technical problems like this is to "fill up" the parameter space of the lowest-dimensional realistic model (in this case 5D), before moving to a higher dimension. As regards other kinds of "charges" associated with particle physics, they should be geometrized and then treated as coordinates in the matching N-dimensional manifold. In this regard, as we have emphasized, there are choices to be made about how best to put the physics into correspondence with the algebra. For example, in supersymmetry, every integral-spin boson is matched with a half-integral-spin fermion, in order to cancel off the enormous vacuum or zero-point fields which would otherwise occur. Now, it is a theorem that any curved energy-full solution of the 4D Einstein field equations can be embedded on a flat and energy-free 10D manifold. (This is basically a result of counting the degrees of freedom in the relevant sets of equations: see Section 4 below). This is the simplest motivation known to the writer for supersymmetry. However, it is possible in certain cases that the condition of zero energy can be accomplished in a space of less than 10 dimensions, given a skillful choice of parameters.

We as physicists have chosen geometry as the currently best way to deal

with macroscopic and microscopic mechanics; and while there are theorems which deal with the question of how to embed the 4D world of our senses in higher-dimensional manifolds, the choice of the latter requires intuition and skill.

3 Eddington and His Legacy

In studying dimensions and fundamental constants over several decades, the writer has come to realize that much modern work on these topics has its roots in the views of Arthur Stanley Eddington (1882-1944; for a recent interdisciplinary review of his contribution to physics and philosophy, see the conference notes edited by Price and French, 2004). He was primarily an astronomer, but with a gift for the pithy quote. For example: "We are bits of stellar matter that got cold by accident, bits of a star gone wrong". However, Eddington also thought deeply about more basic subjects, particularly the way in which science is done, and was of the opinion that much of physics is subjective, insofar as we necessarily filter data about the external world through our human-based senses. Hence the oft-repeated quote: "To put the conclusion crudely-the stuff of the world is mind-stuff". The purpose

of the present section is to give a short and informal account of Eddington's views, and thereby alert workers in fundamental physics to his influence.

This was primarily through a series of non-technical books and his personal contacts with a series of great scientists who followed his lead. These include Dirac, Hoyle and McCrea. In the preceding section, we noted that while it is possible to add an arbitrary number of extra dimensions to relativity as an exercise in mathematics, we need to use the fundamental constants to identify their relevance to physics. (We are here talking primarily about the speed of light c, the gravitational constant G and Planck's constant of action h, which on division by 2π also provides the quantum of spin angular momentum.) To appreciate Eddington's legacy, we note that his writings contain the first logical account of the large dimensionless numbers which occur in cosmology, thereby presaging what Dirac would later formalize as the Large Numbers Hypothesis. This consists basically in the assertion that large numbers of order 10^{40} are in fact equal, which leads among other consequences to the expectation that G is variable over the age of the universe (see Wesson 1978). This possibility is now discussed in the context of field theory in $N > 4$ dimensions, where the dynamics of the higher-dimensional manifold implies that the coupling constants (like G) in 4D are changing functions of

the spacetime coordinates (Wesson 1999). One also finds in Eddington's works some very insightful, if controversial, comments about the so-called fundamental constants. These appear to have influenced Hoyle, who argued that the c^2 in the common relativistic expression $(c^2t^2 - x^2 - y^2 - z^2)$ should not be there, because "there is no more logical reason for using a different time unit than there would be for measuring x, y, z in different units". The same influence seems to have acted on McCrea, who regarded c, G and h as "conversion constants and nothing more". These comments are in agreement with the view advanced in Section 2, namely that the fundamental constants are parameters which can be used to change the physical units of material quantities to lengths, enabling them to be given a geometrical description. There is a corollary of this view which is pertinent to several modern versions of higher-dimensional physics. Whatever the size of the manifold, the equations of the related physics are homogeneous in their physical units (M, L, T) so they can always be regarded as equalities involving <u>dimensionless</u> parameters. It makes sense to consider the possible (say) time variation of such parameters; but it makes no sense to argue that the component <u>dimensionalful</u> quantities are variable. To paraphrase Dicke: Physics basically consists of the comparison of dimensionless parameters at

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different points in the manifold.

Views like this <u>still</u> raise the hackles of certain physicists who have not analysed the problem at a deep level. Eddington, in particular, was severely criticized by both physicists and philosophers when he presented his opinions in the 1930s. Fortunately, many workers - as a result of their studies of unified field theory - came to a sympathetic understanding of Eddington's opinions in the 1990s. However, there is an interesting question of psychology involved here.

Plato tells us of an artisan whose products are the result of experience and skill and meet with the praise of his public for many years. However, in later times he suddenly produces a work which is stridently opposed to tradition and incurs widespread criticism. Has the artisan suffered some delusion, or has he broken through to an art form so novel that his pedestrian-minded customers cannot appreciate or understand it?

Eddington spent the first part of his academic career doing well-regarded research on stars and other aspects of conventional astronomy. He then showed great insight and mathematical ability in his study of the then-new subject of general relativity. In his later years, however, he delved into the arcane topic of the dimensionless numbers of physics, attempt-

ing to derive them from an approach which combined elements of pure reason and mathematics. This approach figures significantly in his book Relativity Theory of Protons and Electrons, and in the much-studied posthumous volume Fundamental Theory. The approach fits naturally into his philosophy of science, which latter argued that many results in physics are the result of how we do science, rather than direct discoveries about the external world (which, however, he admitted). Jeffreys succeeded Eddington to the Plumian Chair at Cambridge, but was a modest man more interested in geophysics and the formation of the solar system than the speculative subject of cosmology. Nevertheless, he developed what at the time was a fundamental approach to the theory of probability, and applied his skills to a statistical analysis of Eddington's results. The conclusion was surprising: according to Jeffreys' analysis of the uncertainties in the underlying data which Eddington had used to construct his account of the basic physical parameters, the results agreed with the data better than they ought to have done. This raised the suspicion that Eddington had "cooked" the results. This author spent the summer of 1970 in Cambridge, having written (during the preceding summer break from undergraduate studies at the University of London) a paper on geophysics which appealed to Jeffreys. We discussed, among

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other things, the status of Eddington's results. Jeffreys had great respect for Eddington's abilities, but was of the opinion that his predecessor had <u>unwittingly</u> put subjective elements into his approach which accounted for their unreasonable degree of perfection. The writer pointed out that there was another possible explanation: that Eddington was in fact right in his belief that the results of physics were derivable from first principles, and that his approach was compatible with a more profound theory which yet awaits discovery.

4 <u>Campbell and His Theorem</u>

Whatever the form of a new theory which unifies gravity with the forces of particle physics, there is a consensus that it will involve extra dimensions. In Section 2, we considered mainly the 5D approach, which by the modern names of induced-matter and membrane theory is essentially old Kaluza-Klein theory without the stifling condition of compactification. The latter, wherein the extra dimension is "rolled up" to a very small size, answers the question of why we do not "see" the fifth dimension. However, an equally valid answer to this is that we are constrained to live close to a hypersurface,

like an observer who walks across the surface of the Earth without being directly aware of what lies beneath his feet. In this interpretation, 5D general relativity must be regarded as a kind of new standard. It is the simplest extension of Einstein's theory, and is widely viewed as the low-energy limit of more sophisticated theories which accommodate the internal symmetry groups of particle physics, as in 10D supersymmetry, 11D supergravity and 26D string theory. There is, though, no sancrosanct value of the dimensionality N. It has to be chosen with a view to what physics is to be explained. (In this regard, St. Kalitzin many years ago considered $N \to \infty$.) All this understood, however, there is a practical issue which needs to be addressed and is common to all higher-N theories: How do we embed a space of dimension N in one of dimension $(N + 1)$? This is of particular relevance to the embedding of 4D Einstein theory in 5D Kaluza-Klein theory. We will consider this issue in the present section, under the rubric of Campbell's theorem. While it is central and apparently simple, it turns out to have a rather long history with some novel implications.

John Edward Campbell was a professor of mathematics at Oxford whose book "A Course of Differential Geometry" was published posthumously in 1926. The book is basically a set of lecture notes on the algebraic properties

of ND Riemannian manifolds, and the question of embeddings is treated in the latter part (notably chapters 12 and 14). However, what is nowadays called Campbell's theorem is there only sketched. He had intended to add a chapter dealing with the relation between abstract spaces and Einstein's theory of general relativity (which was then a recent addition to physics), but died before he could complete it. The book was compiled with the aid of Campbell's colleague, E.B. Elliot, but while accurate is certainly incomplete.

The problem of embedding an ND (pseudo-) Riemannian manifold in a Ricci-flat space of one higher dimension was taken up again by Magaard. He essentially proved the theorem in his Ph.D. thesis of 1963. This and subsequent extensions of the theorem have been discussed by Seahra and Wesson (2003), who start from the Gauss-Codazzi equations and consider an alternative proof which can be applied to the induced-matter and membrane theories mentioned above.

The rediscovery of Campbell's theorem by physicists can be attributed largely to the work of Tavakol and coworkers. They wrote a series of articles in mid-1990s which showed a connection between the CM theorem and a large body of earlier results by Wesson and coworkers (Wesson 1999). The latter group had been using 5D geometry as originally introduced by Kaluza and

Klein to give a firm basis to the aforementioned idea of Einstein, who wished to transpose the "base-wood" of the right-hand side of his field equations into the "marble" of the left-hand side. That an effective or induced 4D energy-momentum tensor $T_{\alpha\beta}$ can be obtained from a 5D geometrical object such as the Ricci Tensor R_{AB} is evident from a consideration of the number of degrees of freedom involved in the problem (see below). The only requirement is that the 5D metric tensor be left general, and not be restricted by artificial constraints such as the "cylinder" condition imposed by Kaluza and Klein. Given a 5D line element $dS^2 = g_{AB}\left(x^\gamma, l\right) dx^A dx^B$ $(A, B = t, xyz, l)$ it is then merely a question of algebra to show that the equations $R_{AB} = 0$ contain the ones $G_{\alpha\beta} = T_{\alpha\beta}$ named after Einstein. (In accordance with comments about the non-fundamental nature of the constants, and common practice, we in this section choose units which render $8\pi G / c^4$ equal to unity.) Many exact solutions of $R_{AB} = 0$ are now known (see Wesson 2006 for a catalog). Of these, special mention should be made of the "standard" 5D cosmological ones due to Ponce de Leon, and the 1-body and other solutions in the "canonical" coordinates introduced by Mashhoon et al. It says something about the divide between physics and mathematics, that the connection between these solutions and the CM theorem was only made later, by the aforementioned

work of Tavakol et al. Incidentally, these workers also pointed out the implications of the CM theorem for lower-dimensional $(N < 4)$ gravity, which some researchers believe to be relevant to the quantization of this force.

The CM theorem, which we will re-prove below, is a <u>local</u> embedding theorem. It cannot be pushed towards solving problems which are the domain of (more difficult) <u>global</u> embeddings. This implies that the CM theorem should not be applied to initial-value problems or situations involving singularities. It is a modest - but still very useful - result, whose main implication is that we can gain a better understanding of matter in 4D by looking at the field equations in 5D.

The CM theorem in succinct form says: Any analytic Riemannian space $V_n (s, t)$ can be locally embedded in a Ricci-flat Riemannian space $V_{n+1} (s + 1, t)$ or $V_{n+1} (s, t + 1)$.

We are here using the convention that the "small" space has dimensionality n with coordinates running 0 to $n - 1$, while the "large" space has dimensionality $n + 1$ with coordinates running 0 to n. The total dimensionality is $N = 1 + n$, and the main physical focus is on $N = 5$.

The CM theorem provides a mathematical basis for the induced-matter theory, wherein matter in 4D as described by Einstein's equations $G_{\alpha\beta} =$

$T_{\alpha\beta}$ is derived from apparent vacuum in 5D as described by the Ricci-flat equations $R_{AB} = 0$ (Wesson 1999, 2006). The main result is that the latter set of relations satisfy the former set if

$$
\begin{aligned}
T_{\alpha\beta} \;=\; & \frac{\Phi_{,\alpha;\beta}}{\Phi} - \frac{\varepsilon}{2\Phi^2} \left\{ \frac{\Phi_{,4} g_{\alpha\beta,4}}{\Phi} - g_{\alpha\beta,44} + g^{\lambda\mu} g_{\alpha\lambda,4} g_{\beta\mu,4} \right. \\
& \left. -\frac{g^{\mu\nu} g_{\mu\nu,4} g_{\alpha\beta,4}}{2} + \frac{g_{\alpha\beta}}{4} \left[g^{\mu\nu}_{,4} g_{\mu\nu,4} + (g^{\mu\nu} g_{\mu\nu,4})^2 \right] \right\} \quad .
\end{aligned}
$$

Here the 5D line element is $dS^2 = g_{\alpha\beta}(x^\gamma, l) dx^\alpha dx^\beta + \varepsilon \Phi^2 (x^\gamma, l) dl^2$, where $\varepsilon = \pm 1$, a comma denotes the ordinary partial derivative and a semicolon denotes the ordinary 4D covariant derivative. Nowadays, it is possible to prove Campbell's theorem using the ADM formalism, whose lapse-and-shift technique has been applied extensively to derive the energy of 5D solutions. It is also possible to elucidate the connection between a smooth 5D manifold (as in induced-matter theory) and one containing a singular surface (as in membrane theory). We now proceed to give an ultra-brief account of this subject.

Consider an arbitrary manifold Σ_n in a Ricci-flat space V_{n+1}. The embedding can be visualized by drawing a line to represent Σ_n in a surface, the normal vector n^A to it satisfying $n \cdot n \equiv n^A n_A = \varepsilon = \pm 1$. If $e^A_{(\alpha)}$ represents an appropriate basis and the extrinsic curvature of Σ_n is $K_{\alpha\beta}$, the ADM

constraints read

$$G_{AB}n^A n^B = -\frac{1}{2}\left(\varepsilon R^\alpha_\alpha + K_{\alpha\beta}K^{\alpha\beta} - K^2\right) = 0$$

$$G_{AB}e^A_{(\alpha)}n^B = K^\beta_{\alpha;\beta} - K_{,\alpha} = 0 \quad .$$

These relations provide $1 + n$ equations for the $2 \times n\left(n+1\right)/2$ quantities $g_{\alpha\beta}$, $K_{\alpha\beta}$. Given an arbitrary geometry $g_{\alpha\beta}$ for Σ_n, the constraints therefore form an under-determined system for $K_{\alpha\beta}$, so infinitely many embeddings are possible.

This demonstration of Campbell's theorem can easily be extended to the case where V_{n+1} is a de Sitter space or anti-de Sitter space with an explicit cosmological constant, as in some applications of brane theory. Depending on the application, the remaining $n\left(n+1\right) - \left(n+1\right) = \left(n^2 - 1\right)$ degrees of freedom may be removed by imposing initial conditions on the geometry, physical conditions on the matter, or conditions on a boundary.

The last is relevant to brane theory with the Z_2 symmetry, where $dS^2 = g_{\alpha\beta}\left(x^\gamma, l\right) dx^\alpha dx^\beta + \varepsilon dl^2$ with $g_{\alpha\beta} = g_{\alpha\beta}\left(x^\gamma, +l\right)$ for $l \geq 0$ and $g_{\alpha\beta} = g_{\alpha\beta}\left(x^\gamma, -l\right)$ for $l \leq 0$ in the bulk. Non-gravitational fields are confined to the brane at $l = 0$, which is a singular surface. Let the energy-momentum in the brane be represented by $\delta\left(l\right) S_{AB}$ (where $S_{AB}n^A = 0$) and that in the bulk by T_{AB}. Then the field equations read $G_{AB} = \kappa\left[\delta\left(l\right) S_{AB} + T_{AB}\right]$ where κ is a 5D

coupling constant. The extrinsic curvature discussed above changes across the brane by an amount $\Delta_{\alpha\beta} \equiv K_{\alpha\beta}(\Sigma_{l>0}) - K_{\alpha\beta}(\Sigma_{l<0})$ which is given by the Israel junction conditions. These imply

$$\Delta_{\alpha\beta} = -\kappa \left(S_{\alpha\beta} - \frac{1}{3} S g_{\alpha\beta} \right) \quad .$$

But the $l = 0$ plane is symmetric, so

$$K_{\alpha\beta}(\Sigma_{l>0}) = -K_{\alpha\beta}(\Sigma_{l<0}) = -\frac{\kappa}{2} \left(S_{\alpha\beta} - \frac{1}{3} S g_{\alpha\beta} \right) \quad .$$

This result can be used to evaluate the 4-tensor

$$P_{\alpha\beta} \equiv K_{\alpha\beta} - K g_{\alpha\beta} = -\frac{\kappa}{2} S_{\alpha\beta} \quad .$$

However, $P_{\alpha\beta}$ is actually identical to the 4-tensor $(g_{\alpha\beta,4} - g_{\alpha\beta} g^{\mu\nu} g_{\mu\nu,4})/2\Phi$ of induced-matter theory, where it figures in 4 of the 15 field equations $R_{AB} = 0$ as $P^{\beta}_{\alpha;\beta} = 0$ (Wesson 1999). That is, the conserved tensor $P_{\alpha\beta}$ of induced-matter theory is essentially the same as the total energy-momentum tensor in Z_2-symmetric brane theory. Other correspondences can be established in a similar fashion.

Thus while induced-matter theory and membrane theory are often presented as alternatives, they are in fact the same thing, and from the viewpoint of differential geometry both are rooted in the CM theorem. This theorem

also has the wider implication that, given the physics in a given manifold, we can always derive the corresponding physics in a manifold of plus-or-minus one dimension. In other words, Campbell's theorem provides a kind of ladder which enables us to go up or down between manifolds of different dimensionality.

5 <u>Summary</u>

Dimensions are a delightful subject with which to dally, but we should remind ourselves that they need the cold scrutiny of common sense to be useful. This means, among other things, that we should have a physical identification of the extra coordinates, in order to understand the implications of their associated dimensions. In 5D, we have seen that the extra coordinate can profitably be related to rest mass, either as the Schwarzschild radius or the Compton wavelength, in the classical and quantum domains respectively. This implies that the fifth dimension is a scalar field, which is presumably the classical analog of the Higgs field by which particles acquire mass in quantum field theory. This interpretation depends on a judicial use of the fundamental constants (Section 2). This approach gives much to the work of

Eddington, who delved deeply into the meanings of the equations of physics (Section 3). Our usage of dimensions also owes something to Campbell, whose theorem in its modern form shows how to go between manifolds whose dimensionality differs by one (Section 3). Our conclusion is that while the use of dimensions may in some respects resemble a game of chess, to be of practical importance we need to ascribe the appropriate physical labels to the coordinates and the spaces, something which requires skill.

Acknowledgements

References

- Barrow, J.D., 1981. Quant. J. Roy. Astron. Soc. <u>22</u>, 388.

- Barrow, J.D., Tipler, F.J., 1986. The Anthropic Principle. Oxford Un. Press, New York.

- Campbell, J.E., 1926. A Course of Differential Geometry. Clarendon Press, Oxford.

- Eddington, A.S., 1935. New Pathways in Science. Cambridge Un.

Press, Cambridge.

- Eddington, A.S., 1939. The Philosophy of Science. Cambridge Un. Press, Cambridge.

- Green, M.B., Schwarz, J.H., Witten, E., 1987. Superstring Theory. Cambridge Un. Press, Cambridge.

- Gubser, S.S., Lykken, J.D., 2004. Strings, Branes and Extra Dimensions. World Scientific, Singapore.

- Halpern, P., 2004. The Great Beyond: Higher Dimensions, Parallel Universes, and the Extraordinary Search for a Theory of Everything. J. Wiley, Hoboken, New Jersey.

- Kilmister, C.W., 1994. Eddinton's Search for a Fundamental Theory. Cambridege University Press, Cambridge.

- McCrea, W.H., Rees, M.J. (eds.), 1983. Phil. Trans. Roy. Soc. (London) A $\underline{310}$, 209.

- Petley, B.W., 1985. The Fundamental Constants and the Frontier of Measurement. Hilger, Bristol.

- Price, K., French, S. (eds.), 2004. Arthur Stanley Eddington: Interdisiplinary Perspectives. Centre for Research in the Arts, Humanities and Social Sciences (10-11 March), Cambridge.

- Seahra, S.S., Wesson, P.S., 2003. Class Quant. Grav. $\underline{20}$, 1321.

- Szabo, R.J., 2004. An Introduction to String Theory and D-Brane Dynamics. World Scientific, Singapore.

- Wesson, P.S., 1978. Cosmology and Geophysics. Hilger/Oxford Un. Press, New York.

- Wesson, P.S., 1992. Space Science Reviews, $\underline{59}$, 365.

- Wesson, P.S., 1999. Space, Time, Matter. World Scientific Publishing Corporation, Singapore.

- Wesson, P.S., 2006. Five-Dimensional Physics. World Scientific, Singapore.

- West, P., 1986. Introduction to Supersymmetry and Supergravity. World Scientific, Singapore.